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INDECOMPOSABLE POLYTOPES

BY

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ABSTRACT

The notions of local similarity and decomposability are extended to the class of geometric graphs. These, in turn, are used to produce new sufficient conditions for indecomposability of polytopes. A simple example is given of two combinatorially equivalent 3-polytopes, one decomposable, and the other not.

1. A convex polytope Q is a *summand* of a convex polytope P if there exists a polytope R such that $P = Q + R$ (i.e., $P = \{x + y : x \in Q, y \in R\}$). The set of summands of a polytope P is convex, and most useful information about it can be obtained from the cone $\mathcal{S}(P) = \{Q : \lambda Q \text{ is a summand of } P \text{ for some } \lambda > 0\}$. A polytope P is *indecomposable* if $\mathcal{S}(P)$ consists of nonnegative homothets of P only. A polytope is *combinatorially indecomposable* if every realization of its combinatorial type is indecomposable.

Indecomposable 2-polytopes are just triangles ([9]), but no similar simple geometric characterization of indecomposable d-polytopes exists for $d > 2$. An algebraic characterization has been given in [4] and [5], but it has no obvious geometric significance. Sufficient conditions, both for decomposability and indecomposability, have been given in [1], [2] (15.1), and [7]. This paper is mainly related to theorem (12) in [7]:

THEOREM 1. (G. C. Shephard). *A convex polytope P is indecomposable if it has an edge E, to which each vertex p of P is indecomposably connected, i.e., for each $p \in \operatorname{vert} P$ there is a chain $F_1, \cdots, F_n$ of indecomposable faces of P, such that $E \subset F_1$, $p \in F_n$, and $\dim F_i \cap F_{i+1} > 0$ for each $1 \leqq i < n$.*

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The main results of this paper are extensions of this theorem. An important tool for their proof is the concept of local similarity. For a nonempty convex polytope in E^d and a unit vector $u \in S^{d-1}$, let $F(P, u)$ denote the face of P which is the intersection of P with a supporting hyperplane with outward normal u. Write $P \leqq Q$ if $\dim F(P,u) \leqq \dim F(Q,u)$ for all $u \in S^{d-1}$ (this definition of "$P \leqq Q$" differs slightly from the original one in [7], and follows W. Meyer's definition in [5]). The polytopes P and Q are *locally similar* ($P \sim Q$) if $P \leqq Q$ and $Q \leqq P$. These notations are used in the following ([7, theorem (4)]):

THEOREM 2 (G. C. Shephard). *If P and Q are convex polytopes, then $Q \in \mathcal{S}(P)$ iff $Q \leqq P$.*

For a nonempty face F of a d-polytope P in E^d let $\sigma(P, F)$ be the (closed) *spherical image* of F, i.e., the set of all vectors $u \in S^{d-1}$ for which $F(P, u) \geqq F$. The *spherical complex $C(P)$ of P* is the set of spherical images of all nonempty faces of P, which forms a spherical complex in the sense of [8]. The image of an n-face is a $d-n-1$ cell in this complex, and the mapping $F \rightarrow \sigma(P, F)$ is an anti-isomorphism between the face-lattice $\mathcal{F}(P) \setminus \{\emptyset\}$ and the complex $C(P)$.

2. The results of this paper require some new terminology. A *geometric graph* (GG) in E^d is a pair $G = (V, E)$, where V ($=\operatorname{vert} G$) is a finite set of points in E^d, called vertices, and E is a set of nondegenerate straight line segments with endpoints in V, called edges. The *union* $A \cup B$ of two GG's, the *intersection* $A \cap B$, an *isomorphism* between A and B, and the notion of a *sub-GG* of A are defined in a natural way, exactly as in the theory of abstract graphs. The GG's of a single point $(\{x\}, \emptyset)$, a line segment $(\{x, y\}, \{[x, y]\})$ and a (possibly degenerate) triangle $(\{x, y, z\}, \{[x, y], [x, z], [y, z]\})$ are denoted in short by (x), (x, y) and (x, y, z), respectively. Two GG's A, B are (positively) *homothetic* if there exists an isomorphism $\varphi : A \rightarrow B$ such that $\varphi(p) = x + \lambda p$ for all $p \in \operatorname{vert} A$, with some fixed $x \in E^d$ and $\lambda > 0$. Such a mapping is called a *homothety.*

An isomorphism $\varphi : A \rightarrow B$ is a *local similarity* if for every edge $[p, q]$ of A, the restriction of φ to the sub-GG (p, q) is a homothety (in other words, if the vector $\varphi(p) - \varphi(q)$ is a positive multiple of $p - q$ for every edge $[p, q]$ of A). A and B are *locally similar* ($A \sim B$) if there is a local similarity between them.

A GG is *indecomposable* if every local similarity between A and any other GG is a homothety. It follows from the definitions that a local similarity between indecomposable GG's is determined by its action on two points. Note that an indecomposable GG is necessarily connected.

The edge graph (= 1 skeleton) of a convex polytope P forms a GG, denoted by $G(P)$.

The following Lemma 3 and Theorem 4 provide the connection between all these concepts.

LEMMA 3. *Let P be a d-polytope, K a* GG, *and* $Q = \operatorname{conv}\operatorname{vert} K$. *If $G(P) \sim K$, then $C(Q) = C(P)$, and $G(Q) = K$.*

PROOF. Let $\varphi : \operatorname{vert} P \to \operatorname{vert} K$ be the local similarity mapping between $G(P)$ and K. If $p \in \operatorname{vert} P$ and $u \in S^{d-1}$, then $u \in \operatorname{relint} \sigma(P, p)$ iff $\langle p, u\rangle > \langle q, u\rangle$ for all $q \in \operatorname{vert} P \setminus \{p\}$. In this case p can be connected to q by an edge path S along which the function $\langle x, u\rangle$ is decreasing ([3, p. 213]). The image $\varphi(S)$ of S under φ is an edge path in K that connects $\varphi(p)$ to $\varphi(q)$, and the function $\{x, u\rangle$ decreases along $\varphi(S)$, because φ is a local similarity. Therefore $\langle \varphi(p), u\rangle > \langle \varphi(q), u\rangle$ for all $q \in \operatorname{vert} P \setminus \{p\}$. Since $\varphi(\operatorname{vert} P) \supseteq \operatorname{vert} Q$, it follows that $\langle \varphi(p), u\rangle > \langle x, u\rangle$ for all $x \in Q \setminus \{\varphi(p)\}$, and therefore $\varphi(p) \in \operatorname{vert} Q$ and $u \in \operatorname{relint} \sigma(Q, \varphi(p))$.

If $u \notin \operatorname{relint} \sigma(P, p)$ then $\langle p, u\rangle \leqq \langle q, u\rangle$ for some $q \in \operatorname{vert} P$. Following a similar argument we obtain $\langle \varphi(p), u\rangle \leqq \langle \varphi(q), u\rangle$, and therefore $u \notin \operatorname{relint} \sigma(Q, \varphi(p))$. Thus $\operatorname{vert} Q = \varphi(\operatorname{vert} P)$ and $\sigma(P, p) = \sigma(Q, \varphi(p))$ for all $p \in \operatorname{vert} P$.

So far we have seen that the spherical complexes $C(P)$ and $C(Q)$ have exactly the same $(d-1)$-cells. Since every cell of $C(P)$ and $C(Q)$ is an intersection of $(d-1)$-cells, we conclude that $C(P) = C(Q)$. From $C(P) = C(Q)$ it follows that $G(P) \sim G(Q)$, and therefore $G(Q) \sim K$. But $G(Q)$ and K have the same vertices, hence $G(Q) = K$.

THEOREM 4. *Let P and Q be two d-polytopes in E^d. The following conditions are equivalent:*

(a) $P \sim Q$,

(b) $P \in \mathscr{S}(Q)$ *and* $Q \in \mathscr{S}(P)$,

(c) $C(P) = C(Q)$,

(d) $G(P) \sim G(Q)$,

(e) $Q \in \operatorname{relint} \mathscr{S}(P)$.

PROOF. The equivalence (a) $\Leftrightarrow$ (b) is an immediate consequence of Theorem 2 and the definition of local similarity.

For a unit vector $u \in S^{d-1}$ denote by $C(P, u)$ $(C(Q, u))$ the unique cell of $C(P)$ $(C(Q))$ which contains u in its relative interior. The implication (c) $\Rightarrow$ (a) follows readily from the equality $\dim F(P, u) + \dim C(P, u) = d - 1$. To establish the equivalence (a) $\Leftrightarrow$ (c) (which was, in fact, proved in [4] as Corollary 2.7), assume that $C(P) \neq C(Q)$. Since every cell of $C(P)$ and $C(Q)$ is an intersection of $(d-1)$-cells, the inequality $C(P) \neq C(Q)$ implies that for some $u \in S^{d-1}$, $C(P, u) \neq C(Q, u)$ and $\dim C(P, u) = \dim C(Q, u) = d-1$. Suppose, say, that $C(Q, u) \not\subseteq C(P, u)$, hence $\operatorname{bd} C(P, u)$ intersects $\operatorname{int} C(Q, u)$. Choose a point $w \in \operatorname{bd} C(P, u) \cap \operatorname{int} C(Q, u)$. Then $\dim C(P, w) < d - 1 = \dim C(Q, w)$, hence $\dim F(P, w) > \dim F(Q, w)$, contrary to (a). Thus we have (a) $\Leftrightarrow$ (c).

The proof of the implication (c) $\Rightarrow$ (d) is straightforward, and we omit it. The inverse implication (d) $\Rightarrow$ (c) follows immediately from Lemma 3. For the proof of (b) $\Rightarrow$ (e) we shall first show that $P \in \operatorname{relint} \mathcal{S}(P)$. For any $A \in \mathcal{S}(P)$ there is a positive λ such that λA is a summand of P, i.e., there exists a polytope B such that $\lambda A + B = P$. But this implies

$$P = \frac{\lambda}{\lambda+1} A + \frac{1}{\lambda+1}(B + \lambda P), \qquad \text{where } \frac{\lambda}{\lambda+1} + \frac{1}{\lambda+1} = 1, \quad 1 > \frac{\lambda}{\lambda+1} > 0,$$

and so $B + \lambda P \in \mathcal{S}(P)$. Thus we see that for every $A \in \mathcal{S}(P)$, P is an interior point of a line segment in $\mathcal{S}(P)$ with endpoint A, hence $P \in \operatorname{relint} \mathcal{S}(P)$.

Now $Q \in \mathcal{S}(P)$ implies $\mathcal{S}(Q) \subseteq \mathcal{S}(P)$, because μR is a summand of Q and λQ is a summand of P, then $\lambda \mu R$ is a summand of P. Hence (b) implies $\mathcal{S}(P) = \mathcal{S}(Q)$, and $Q \in \operatorname{relint} \mathcal{S}(Q) = \operatorname{relint} \mathcal{S}(P)$.

Finally we show that (e) $\Rightarrow$ (b). Let Q be a polytope in $\operatorname{relint} \mathcal{S}(P)$. If $Q = P$ then (b) is obviously true. Otherwise, the line segment $[P, Q]$ can be slightly extended in $\mathcal{S}(P)$ to a line segment $[P, R]$, which contains Q as an interior point. Thus λP is a summand of Q for some $\lambda > 0$, hence $P \in \mathcal{S}(Q)$.

Corollary 5. *A polytope P is decomposable if and only if $G(P)$ is decomposable.*

Proof. (1) If P is decomposable then $\operatorname{relint} \mathcal{S}(P)$ contains a polytope Q which is not homothetic to P. Clearly $G(P)$ is not homothetic to $G(Q)$, but by (e) $\Rightarrow$ (d) we have $G(P) \sim G(Q)$, hence $G(P)$ is decomposable.

(2) If $G(P)$ is decomposable, then there exists a GG K such that $K \sim G(P)$ but K is not homothetic to $G(P)$. By Lemma 3, $Q = \operatorname{conv} \operatorname{vert} K$ satisfies $G(Q) = K \sim G(P)$.

By (d) $\Rightarrow$ (b) we have $Q \in \mathcal{S}(P)$, but Q is not a homothet of P, because $G(P)$ and $G(Q)$ are not homothetic. Thus P is decomposable.

3. Theorem 1 may now be regarded as a consequence of the last corollary, and the following two simple lemmas:

LEMMA 6. *A* GG *is indecomposable iff it has an indecomposable sub*-GG L *such that* vert L = vert K.

LEMMA 7. *If two indecomposable* GG's A *and* B *have (at least) a common edge, then* $A \cup B$ *is indecomposable.*

Lemma 6 follows immediately from the definitions, and Lemma 7 is a special case of Theorem 8 below.

Our aim in this section is to establish four theorems of the same type as Lemma 7. These theorems, together with Lemma 6, may serve to establish indecomposability of polytopes, and to obtain results similar to Theorem 1.

THEOREM 8. *If two indecomposable* GG's A *and* B *have (at least) two vertices in common, then* $A \cup B$ *is indecomposable.*

PROOF. Let $\varphi : A \cup B \to C$ be a local similarity. The indecomposability of A and B implies that the restrictions $\varphi_{|A}$ and $\varphi_{|B}$ are homotheties. Since these two homotheties act identically on the vertices of A and B, they are restrictions to vert A, resp. vert B, of one and the same homothety of E^d. Hence φ is a homothety, and this proves that $A \cup B$ is indecomposable.

THEOREM 9. *If* A_1, A_2, A_3 *are three indecomposable* GG's, *and* a_{12}, a_{23}, a_{13} *are three non-collinear distinct vertices, such that* $a_{ij} \in A_i \cap A_j$ *for* $1 \leqq i < j \leqq 3$, *then* $B = A_1 \cup A_2 \cup A_3$ *is indecomposable.*

PROOF. Let $\varphi : B \to C$ be a local similarity. The indecomposability of A_i implies that the restriction $\varphi_{|A_i}$ is a homothety for $1 \leqq i \leqq 3$. Since every two of the three points a_{12}, a_{23}, a_{13} are contained in some A_i, the restriction of φ to the triangle $T = (a_{12}, a_{23}, a_{13})$ is a local similarity, and therefore a homothety, since T is indecomposable. Since T has two common vertices with each A_i, it follows that the four maps $\varphi_{|T}, \varphi_{|A_1}, \varphi_{|A_2}, \varphi_{|A_3}$ are restrictions of one and the same homothety. This proves the indecomposability of B.

THEOREM 10. *Let* C *be the union of two indecomposable* GG's A *and* B *and two disjoint edges* (a_i, b_i), $i = 1,2$, *each connecting* A *to* B (*i.e.*, $a_i \in$ vert A, $b_i \in$ vert B, *and the four points* a_1, a_2, b_1, b_2 *are distinct*). *If the two lines* aff(a_1, b_1) *and* aff (a_2, b_2) *are skew then* C *is indecomposable.*

PROOF. Let $\varphi : C \to D$ be a local similarity. The indecomposability of A and B implies that $\varphi_{|A}$ and $\varphi_{|B}$ are homotheties, and so are, of course, also the restrictions $\varphi_{|(a_1,b_1)}$ and $\varphi_{|(a_2,b_2)}$.

It remains to show that all these homotheties are restrictions of one and the same homothety of E^d. We shall assume, without loss of generality, that the homothety $\varphi_{|A}$ is the identity, and then prove the same about the three other homotheties.

Now $\varphi(a_i) = a_i$ and $\varphi_{|(a_i,b_i)}$ is a homothety, hence $\varphi(b_i) \in \mathrm{aff}(a_i, b_i)$ for $i = 1,2$. Since $\varphi_{|B}$ is a homothety, the line segments $[b_1, b_2]$ and $\varphi([b_1, b_2])$ are parallel or collinear. It follows that $\varphi(b_i) = b_i$, otherwise the two lines $\mathrm{aff}(a_i, b_i)$ would be coplanar or coincide, which contradicts our assumption. Therefore $\varphi_{|B}$ is the identity.

THEOREM 11. *Let C be the union of two indecomposable* GG's *A and B, and three vertex disjoint edges (a_i, b_i), $i = 1,2,3$, each connecting A to B (i.e., a_i is a vertex of A, b_i a vertex of B, and the six points $a_1, a_2, a_3, b_1, b_2, b_3$ are all distinct). If the three projective lines* $\mathrm{lin}(a_i, b_i)$ *are not concurrent, then C is indecomposable.*

PROOF. If any two edges (a_i, b_i) and (a_j, b_j) lie on skew lines, then C is indecomposable according to Theorem 10.

If every two of the three edges (a_i, b_i) are coplanar, then they must all lie in one plane, otherwise the three lines $\mathrm{lin}(a_i, b_i)$ would be concurrent, contradictory to our assumption. Assume, therefore, that the lines are coplanar. Let $\varphi : C \to D$ be a local similarity. The indecomposability of A and B implies that $\varphi_{|A}$ and $\varphi_{|B}$ are homotheties, and so are, of course, also the restrictions $\varphi_{|(a_i,b_i)}$ for $i = 1,2,3$. It remains to show that all these homotheties are restrictions of one and the same homothety of E^d.

We shall assume, without loss of generality, that the homothety $\varphi_{|A}$ is the identity, and then prove the same about the other homotheties.

The three-point GG's (b_1, b_2, b_3) and $(\varphi(b_1), \varphi(b_2), \varphi(b_3))$ are homothetic by $\varphi_{|B}$; hence the inequality $\varphi(b_i) \neq b_i$ for all $1 \leqq i \leqq 3$ would imply that the three lines $\mathrm{lin}(b_i, \varphi(b_i))$ all contain the fixed point of the homothety $\varphi_{|B}$, which may be at infinity in case $\varphi_{|B}$ is a translation. (If (b_1, b_2, b_3) is a proper triangle then the concurrence of the three lines also follows from Desargue's Theorem [6, p. 6].) But $\mathrm{lin}(b_i, \varphi(b_i)) = \mathrm{lin}(a_i, b_i)$, and this would contradict our assumption.

If $\varphi(b_k) = b_k$ for one k only, then for any $j \neq k$, the two distinct points b_i and $\varphi(b_i)$ are common to the lines $\mathrm{lin}(b_k, b_j)$ and $\mathrm{lin}(a_j, b_j)$, which therefore coincide. Hence the point b_k lies on all three lines $\mathrm{lin}(a_i, b_i)$, which again contradicts our hypothesis. Hence $\varphi(b_i) = b_i$ for at least two values of i, and $\varphi_{|B}$ is the identity, since B is indecomposable. This, in turn, implies that φ is the identity, which completes the proof.

4. Theorem 1 can now be extended in two directions:

THEOREM 1a. *A polytope P is indecomposable if every two vertices of P belong to some indecomposable sub-*GG *of* $G(P)$.

REMARK. Note that unlike Theorem 1, this version does not require a fixed "anchor" edge E to which every vertex is indecomposably connected.

PROOF. Let $[p, q]$ be an edge of P, and let G be a maximal indecomposable sub-GG of $G(P)$ which includes the GG (p, q). Let x be any vertex of P, $x \neq p, q$. By the assumption of the theorem x is a vertex of two indecomposable sub-GG's K and L of $G(P)$ with $p \in \operatorname{vert} K$ and $q \in \operatorname{vert} L$. By Theorem 9, $G \cup K \cup L$ is indecomposable, hence the maximality of G implies $x \in G$. This proves that $\operatorname{vert} G = \operatorname{vert} P$, and P is indecomposable, by Lemma 6 and Corollary 5.

THEOREM 1b. *A d-polytope P ($d \geqq 3$) is indecomposable if $G(P)$ has an indecomposable sub-*GG *K such that all the components of $G(P) \setminus K$ are isolated vertices or isolated edges.*

PROOF. Let G be a maximal indecomposable sub-GG of $G(P)$ which includes K, and let x be any vertex of P. Suppose $x \notin \operatorname{vert} G$. The component C of $G(P) \setminus G$ which contains x is either the GG (x), or the GG (x, y) for some y, where $[x, y]$ is an edge of P. If $C = (x)$, then P has at least two different edges $[x, w]$ and $[x, z]$, with w and z in $\operatorname{vert} G$. By Theorem 9, $G \cup (x, w) \cup (x, z)$ is indecomposable, which contradicts the maximality of G. If $C = (x, y)$ then each one of the vertices x and y is connected to G by at least two edges of P. From among these edges we can choose a pair of edges, say $[x, z]$ and $[y, w]$, which lie on skew lines. By Theorem 10, $G \cup (x, z) \cup (y, w) \cup (x, y)$ is indecomposable, which again contradicts the maximality of G. It follows that $G = G(P)$, and P is indecomposable, as in the proof of Theorem 1a.

5. The following is a simple example of two combinatorially equivalent 3-polytopes, one decomposable and the other not. Another example has previously been given in [4], but this one is simpler.

The decomposable polytope P (see Fig. 1) is obtained by attaching square pyramids to the top and bottom facets of a cube. To obtain the indecomposable version Q, tilt one of the "walls" of the cube slightly about its bottom edge before attaching the (rectangular) pyramid on the top facet.

P is decomposable, since it is the sum of a vertical line segment and an octahedron. Q is indecomposable by Theorem 10. Let A and B be the indecomposable GG's of the top and bottom pyramids, respectively, and let (a_i, b_i), $i = 1, 2$, be two opposite side edges of the (deformed) cube, which do not

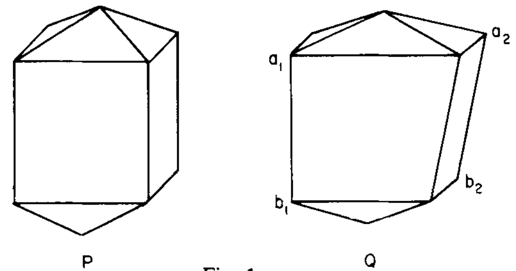

Fig. 1.

lie in the same "wall". The union of A, B and the two edges (a_i, b_i) is a GG which covers vert Q. The two edges (a_i, b_i) lie on skew lines, hence Q is indecomposable.

The indecomposability of Q can be easily proved directly without using Theorem 10. However, it is interesting to see that the skewness of the lines $\mathrm{aff}(a_i, b_i)$ is essential for the indecomposability of Q.

6. In the next example Theorem 11 serves to prove the combinatorial decomposability of a polytope P. In this case there seems to be no obvious alternative proof.

To construct P, start with a regular hexagon h with vertices $a_1, a_2, a_3, a_4, a_5, a_6$ (in this cyclic order). Let K be a bipyramid based on H with apices b, c. Truncate K by two planes H_b, H_c, parallel to $\mathrm{aff}\, H$, that separate h from b and c, respectively. The resulting truncated bipyramid T has 18 vertices a_i, b_i, c_i where $\{b_i\} = H_b \cap [a_i, b]$ and $\{c_i\} = H_i \cap [a_i, c]$ $(1 \leqq i \leqq 6)$, and two hexagonal faces $[b_1, \cdots, b_6]$, $[c_1, \cdots, c_6]$. Next truncate T at b_i down to $a_i, \frac{2}{3}b_i + \frac{1}{3}b_{i+1}, \frac{2}{3}b_i + \frac{1}{3}b_{i-1}$ for $i = 1, 3, 5$ and truncate T in a similar way at c_i for $i = 2, 4, 6$. (Here, of course, $b_0 = b_6$ and $c_7 = c_1$.) The resulting polytope Q has two nonagonal bases B, C, parallel to the hexagonal equator H. Finally, to obtain the desired polytope P, attach to Q two sufficiently flat nonagonal pyramids at the bases B and C.

Figures 2 and 3 give a top view and a side view of P. The top pyramid, together with the three triangles connecting it to the equator H, is clearly an indecompos-

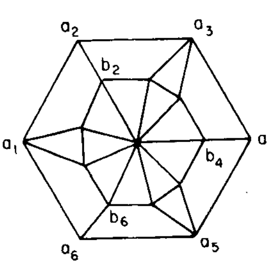

Fig. 2.

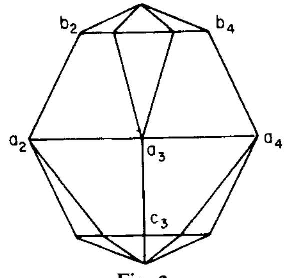

Fig. 3.

able GG, call it A, and so is the GG B consisting of the bottom pyramid with the triangles connecting it to the equator H. A is connected to B by the six edges of H. No three of these edges lie on concurrent lines, and A and B together cover vert P. $G(P)$ is therefore indecomposable by Theorem 11 (and Lemma 6), and thus P is indecomposable.

Let R be any polytope combinatorially equivalent to P. If every two edges of the equator of P are coplanar then every three adjacent edges of the equator are coplanar, and the whole equator of R is a planar hexagon. In this case Q is indecomposable by the same argument as above. If the equator of R has two skew edges then $G(R)$ is indecomposable by Theorem 10 (and Lemma 6).

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